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DYNAMICS OF ELASTIC FOUNDATIONS AND CONSTRAINED BARS

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A variational principle is used to derive the equations of motion of a slender bar that has one lateral face clamped (or welded) onto a rigid surface. Correction factors for the transverse shear and normal strains are incorporated into the expression for the strain energy of the system. These factors are evaluated by requiring the lowest depth-shear and depth-stretch frequencies, as derived from the new approximate theory, to agree with the corresponding frequencies obtained from the full theory of generalized plane stress. The

(See Reverse Side)

20. frequency equation for a simple supported constrained bar is derived, and plots of the variations of the natural frequencies of the transverse and longitudinal modes versus the length to depth ratio of the bar are presented.

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1. INTRODUCTION

Several authors [1] - [5] have discussed methods of deriving the equation of motion of a generalized elastic foundation, which often is treated as a thin elastic layer of thickness h that has one face bonded to the surface of a rigid substrate. Such models, which include the effects of foundation inertia and shear deformation, provide more realistic descriptions of the dynamic response of elastic foundations than does the classical Winkler model, which treats the normal stress at the surface of the foundation as linearly proportional to its transverse deflection w, i.e.,

$$p_2 = -kw \tag{1}$$

However, upon examining the simplest foundation models proposed in references [1], [2], and [4] in the case of slender beams on elastic foundations, it becomes evident that these theories do not yield the correct lowest frequencies for the depth-shear and depth-stretch modes of free vibration. (The depth-shear and depth-stretch modes considered

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[1]V. Z. Vlasov and U. N. Leont'ev. Beams, Plates and Shells on Elastic Foundations. NASA TT F-357 (1966)

[2]N. S. V. Kameswara Rao, Y. C. Das, and M. Anandakrishnan. Variational approach to beams on elastic foundations. Proc. Amer. Soc. Civ. Engrs., J. Engng. Mech. Div., 97, pp. 271-294 (1971).

[3]E. H. Dowell, Dynamic analysis of an elastic plate on a thin, elastic foundation. J. Sound Vib., 35, pp. 343-360 (1974).

[4]N. S. V. Kameswara Rao, Y. C. Das, and M. Anandakrishnan. Dynamic response of beams on generalized elastic foundations. Int. J. Solids Structures, 11, pp. 255-273 (1975).

[5]G. L. Anderson. The influence of a Wieghardt type elastic foundation on the stability of some beams subjected to distributed tangential forces. J. Sound Vib. (to be published).

here are quite analogous to the well-known thickness-shear and thickness-stretch modes that are known to exist in infinite elastic plates.

Related or analogous mathematical problems arise in the design of ultrasonic devices which contain crystal bars and plates that are rigidly clamped on one face due to the mode of mounting.

The objective of this investigation, therefore, consists of determining a system of partial differential equations of motion for thin elastic bars (or foundations) that are rigidly clamped on an entire lateral surface. Such bars will hereinafter be called "constrained bars." These equations may be interpreted as describing the motion of (i) the type of elastic foundation described in the opening paragraph of this section or (ii) an isotropic crystal bar mounted in an ultrasonic device. A variational principle is used to obtain the field equations for the system. The normal and shear strain components of the strain energy of the system are adjusted by means of the introduction of correction factors of the type that are used in the Timoshenko and Mindlin theories for elastic beams and plates, respectively. First order and second order theories are derived, and the various correction factors are evaluated by requiring the lowest depth-shear and depth-stretch frequencies, as derived from the approximate theories, to be identical to those obtained from the full theory of generalized plane stress (see Section 3). Finally, the frequency equation for a simply supported constrained bar of length ℓ is derived, and plots of the variations of the first eight frequencies of the transverse and longitudinal modes versus the length to depth ratio are presented and discussed.

2. GENERALIZED PLANE STRESS

Consider the bar of thickness b in the x_3 - direction ($b << \ell$, where ℓ denotes a characteristic length of the bar) that is depicted in Figure 1. The bar is assumed to be clamped to a rigid surface on the face $x_2 = +h/2$. This implies that the longitudinal and transverse displacements $u_1 (x_1, x_2, t)$ and $u_2 (x_1, x_2, t)$, respectively, are constrained so as to vanish at the surface $x_2 = h/2$, i.e.,

$$u_\alpha (x_1, h/2, t) = 0, \alpha = 1, 2. \tag{2}$$

Furthermore, the lower face of the bar is free of applied surface traction and its longitudinal and transverse displacements are completely unconstrained. Thus, the boundary conditions on this face are those of vanishing shear stress and normal stress, namely,

$$\sigma_{12} = \sigma_{22} = 0 \text{ on } x_2 = -h/2. \tag{3}$$

The equations of motion, according to the theory of generalized plane-stress [6], for the in-plane deformations of a thin, elastic plate are

$$\sigma_{\alpha\beta,\beta} = \rho\ddot{u}_\alpha, \quad \alpha,\beta = 1, 2, \tag{4}$$

where ρ denotes the density of the material, $\sigma_{\alpha\beta,\beta} = \partial\sigma_{\alpha\beta}/\partial x_\beta$, and summation over repeated indices is implied. The constitutive equation is

$$\sigma_{\alpha\beta} = 2\mu\varepsilon_{\alpha\beta} + \gamma\varepsilon_{\sigma\sigma}\delta_{\alpha\beta}, \tag{5}$$

where

$$\gamma = 2\lambda\mu/(\lambda + 2\mu), \tag{6}$$

[6] I.S. Sokolnikoff. Mathematical Theory of Elasticity. New York: McGraw-Hill Publishing Co (1956).

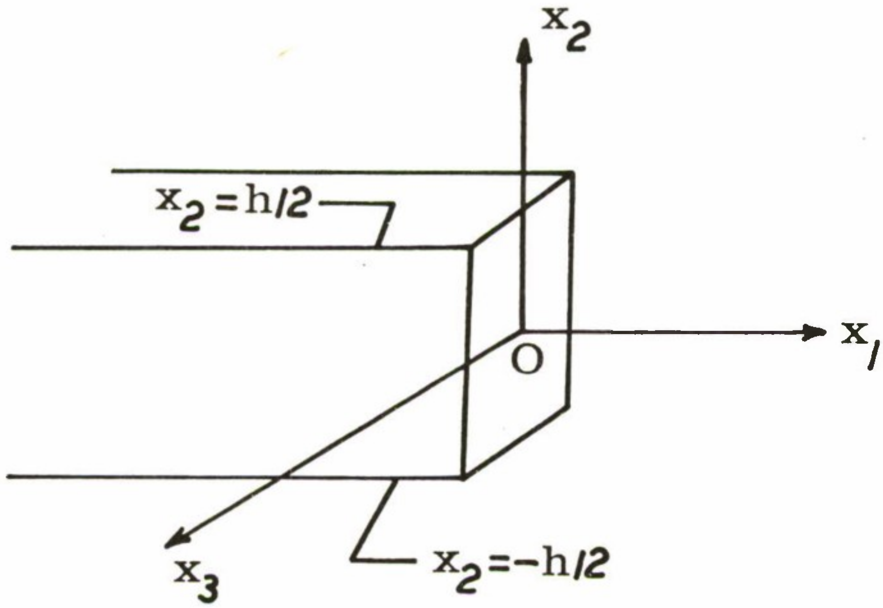

Figure 1. Coordinate system for the bar.

with μ and λ denoting the Lamé constants. The strains $\varepsilon_{\alpha\beta}$ are defined by

$$\varepsilon_{\alpha\beta} = 1/2\ (u_{\alpha,\beta} + u_{\beta,\alpha}). \tag{7}$$

It is easy to verify that the above field equations can be derived directly from a generalized form of Hamilton's principle due to Yu [7], (see also [8]), linearized and specialized here for the case of generalized plane stress. This principle is expressed as

$$\delta \int_{t_0}^{t_i} L dt = 0, \tag{8}$$

where

$$L = T-V+W \tag{9}$$

is the generalized Lagrangian with

$$T = \int_A 1/2\ \rho \dot{u}_\alpha \dot{u}_\alpha\ dA, \qquad dA = dx_1 dx_2, \tag{10}$$

$$V = \int_A (\sigma_{\alpha\beta}\ \varepsilon_{\alpha\beta} - \sigma_{\alpha\beta} e_{\alpha\beta} + U)\ dA, \tag{11}$$

$$W \int_{C_\sigma} \bar{p}_\alpha\ u_\alpha\ ds + \int_{C_u} p_\alpha (u_\alpha - \bar{u}_\alpha)\ ds. \tag{12}$$

Here the $e_{\alpha\beta}$ are generalized strains that will be described later, U, defined by

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[7] Y.-Y. Yu. Generalized Hamilton's principle and variational equation of motion in nonlinear elasticity theory, with application to plate theory. J. Acoust. Soc. Amer., 36, pp. 111-120 (1964).

[8] G. L. Anderson. The stability and non-linear vibrations of an orthotropic hinged-hinged beam. Watervliet Arsenal Technical Report WVT-TR-75036 (1975).

$$U = 1/2\ \sigma_{\alpha\beta}\ e_{\alpha\beta}, \tag{13}$$

is the strain energy density, which is assumed to exist and is a function of $e_{\alpha\beta}$, and $p_\alpha = \sigma_{\alpha\beta}\ n_\beta$, where n_β are the components of the unit normal vector to the boundary of the area A. Moreover, t_0 and t_1 are two instants of time, $\dot{u}_\alpha = \partial u_\alpha/\partial t$, and a barred quantity is a prescribed function. C_σ denotes that portion of the boundary curve C on which the tractions $\bar{p}_\alpha$ are prescribed, C_u that portion of C on which the displacements $\bar{u}_\alpha$ are prescribed, and s denotes arc length along C. Because the variations δu_α, $\delta e_{\alpha\beta}$, $\delta\sigma_{\alpha\beta}$ are arbitrary throughout the area A of the body, δu_α arbitrary on C_σ, and δp_i arbitrary on C_u, their coefficients in the varied form of equation (8) must vanish. Therefore, the present form of Hamilton's principle will generate the stress equations of motion, the stress-strain relations, the strain-displacement relations, the stress boundary conditions, and the displacement boundary conditions. As will become apparent in subsequent sections, this process will be quite convenient for the derivation of the field equations for constrained bars.

3. DEPTH MODES IN AN INFINITE PLATE STRIP

The equation of motion in equation (4) subject to the boundary conditions in equations (2) and (3) admit some rather elementary solutions, herein called depth-shear and depth-stretch modes, in the case of a plate that extends to positive and negative infinity in the x_1-direction. The lowest frequencies of these modes will be of considerable importance later in Section 5.

The depth modes shall be characterized by a displacement field that is independent of the axial coordinate x_1, i.e.,

$$u_\alpha = u_\alpha\,(x_2,\ t). \tag{14}$$

Therefore, in view of equations (14), (5), and (7), the stress equations of motion in equation (4) become simply

$$\mu u_{1,22} = \rho \ddot{u}_1, \qquad |x_2| < h/2 \tag{15}$$

and

$$c_1 u_{2,22} = \rho \ddot{u}_2, \qquad |x_2| < h/2, \tag{16}$$

where

$$c_1 = 4\mu(\mu+\lambda)/(2\mu+\lambda), \tag{17}$$

for the depth-shear and depth-stretch modes, respectively. Moreover, by virtue of equations (2) and (3), the associated boundary conditions are found to be

$$u_{\alpha,2} = 0 \text{ on } x_2 = -h/2,\ u_\alpha = 0 \text{ on } x_2 = h/2 \tag{18}$$

Assuming solutions in the form

$$u_\alpha(x_2,\ t) = v_\alpha(x_2)\cos\omega t, \tag{19}$$

where ω designates the natural circular frequency of vibration of the infinite, constrained plate strip, one obtains from equations (15), (16), and (18), the basic boundary value problem

$$v_{\alpha,22}\,(x_2) + \beta_\alpha^2\, v_\alpha(x_2) = 0 \text{ (no sum on } \alpha), \tag{20}$$

$$v_{\alpha,2}\,(-h/2) = v_\alpha(h/2) = 0, \tag{21}$$

where

$$\beta_1^2 = \rho\omega^2/\mu, \qquad \beta_2^2 = \rho\omega^2/c_1. \tag{22}$$

The solution of equation (20) is

$$v_\alpha(x_2) = A_\alpha \cos\beta_\alpha x_2 + B_\alpha \sin\beta_\alpha x_2. \tag{23}$$

Substitution of this expression into the boundary conditions in equation (21) yields

$$\left.\begin{aligned} A_\alpha \cos \beta_\alpha h/2 + B_\alpha \sin \beta_\alpha h/2 = 0, \\ A_\alpha \sin \beta_\alpha h/2 + B_\alpha \cos \beta_\alpha h/2 = 0. \end{aligned}\right\} \tag{24}$$

The system of homogenous equations (24) will have a non-trivial solution if and only if the coefficient matrix vanishes. This requirement leads to

$$\cos\beta_\alpha h = 0, \tag{25}$$

whence

$$\beta_{\alpha n} = (2n-1)\pi/2h, \quad n=1,2,3,... \tag{26}$$

Therefore, from equations (22) and (26), one finds

$$\omega_n = \frac{(2n-1)\pi}{2h}\sqrt{\frac{\mu}{\rho}}$$

and

$$\omega_n = \frac{(2n-1)\pi}{2h}\sqrt{\frac{c_1}{\rho}}$$

for the depth-shear and depth-stretch modes, respectively. Of particular importance later will be the lowest depth-shear frequency.

$$\omega_1 = \frac{\pi}{2h}\sqrt{\frac{\mu}{\rho}} \tag{27}$$

and the lowest depth-stretch frequency

$$\omega_1 = \frac{\pi}{2h}\sqrt{\frac{c_1}{\rho}} \tag{28}$$

Finally, it may be noted that

$$A_\alpha = \sin \beta_\alpha h/2, \qquad B_\alpha = -\cos\beta_\alpha h/2$$

represents a possible solution of equation (24). Consequently, equation (23) becomes, by virtue of equation (26),

$$v_{\alpha n}(x_2) = \sin\,[(2n-1)\,\pi\,(h-2x_2)/4h]. \tag{29}$$

Naturally, these are the eigenfunctions of equations (20) and (21).

4. A THEORY FOR CONSTRAINED BARS

To develop a theory for constrained bars, one can proceed with the variational principle in equation (8) in much the same way as Vlasov and Leont'ev did in their book [1]. Specifically, it is assumed here that the displacements u_α can be approximated by

$$\left.\begin{aligned} u_1(x_1,x_2,t) &= f_1(x_2)\,u\,(x_1,t) + f_2(x_2)\psi\,(x_1,t), \\ u_2(x_1,x_2,t) &= f_1(x_2)\,w\,(x_1,t) + f_2(x_2)\,\phi(x_1,t), \end{aligned}\right\} \tag{30}$$

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[1] V. Z. Vlasov and U. N. Leont'ev. Beams, Plates and Shells on Elastic Foundations. NASA TT F-357 (1966)

where the linearly independent coordinate functions $f_\alpha(x_2)$ are required to satisfy the geometric boundary condition

$$f_\alpha (h/2) = 0, \qquad \alpha = 1,2, \tag{31}$$

which assures that equation (2) will be satisfied. One possible set of coordinate functions, whose elements satisfy equation (31), is

$$f_1(x_2) = \frac{1}{2h}(h-2x_2), \qquad f_2(x_2) = \frac{-1}{2h^2}(h-2x_2)(h+4x_2). \tag{32}$$

The function $f_1(x_2)$ has previously been employed in references [1] to [4]. Of course, other sets of functions could be selected and, perhaps, would be even more desirable. Further comments on this point appear in Section 9. It may be noted that the functions $f_\alpha(x_2)$ given in equation (32) are orthogonal over the interval $-h/2<x_2<h/2$, i.e.,

$$\int_{-h/2}^{h/2} f_1(x_2)f_2(x_2)dx_2 = 0 \tag{33}$$

and are normalized such that $f_\alpha(-h/2)=1$. The functions f_α have been plotted as functions of $x = 2x_2/h$ in Figure 2. Clearly, $f_1(x_2)$ has a single zero, whereas $f_2(x_2)$ has two zeros.

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[1] V. Z. Vlasov and U. N. Leont'ev. Beams, Plates and Shells on Elastic Foundations. NASA TT F-357 (1966).

[2] N.S.V. Kameswara Rao, Y.C. Das, and M. Anandakrishnan. Variational approach to beams on elastic foundations. Proc. Amer. Soc. Civ. Engrs., J. Engng. Mech. Div., 97, pp. 271-294 (1971).

[3] E. H. Dowell. Dynamic analysis of an elastic plate on a thin, elastic foundation. J. Sound Vib., 35, pp. 343-360 (1974).

[4] N.S.V. Kameswara Rao, Y.C. Das, and M. Anandakrishnan. Dynamic response of beams on generalized elastic foundations. Int. J. Solids Structures, 11, pp. 255-273 (1975).

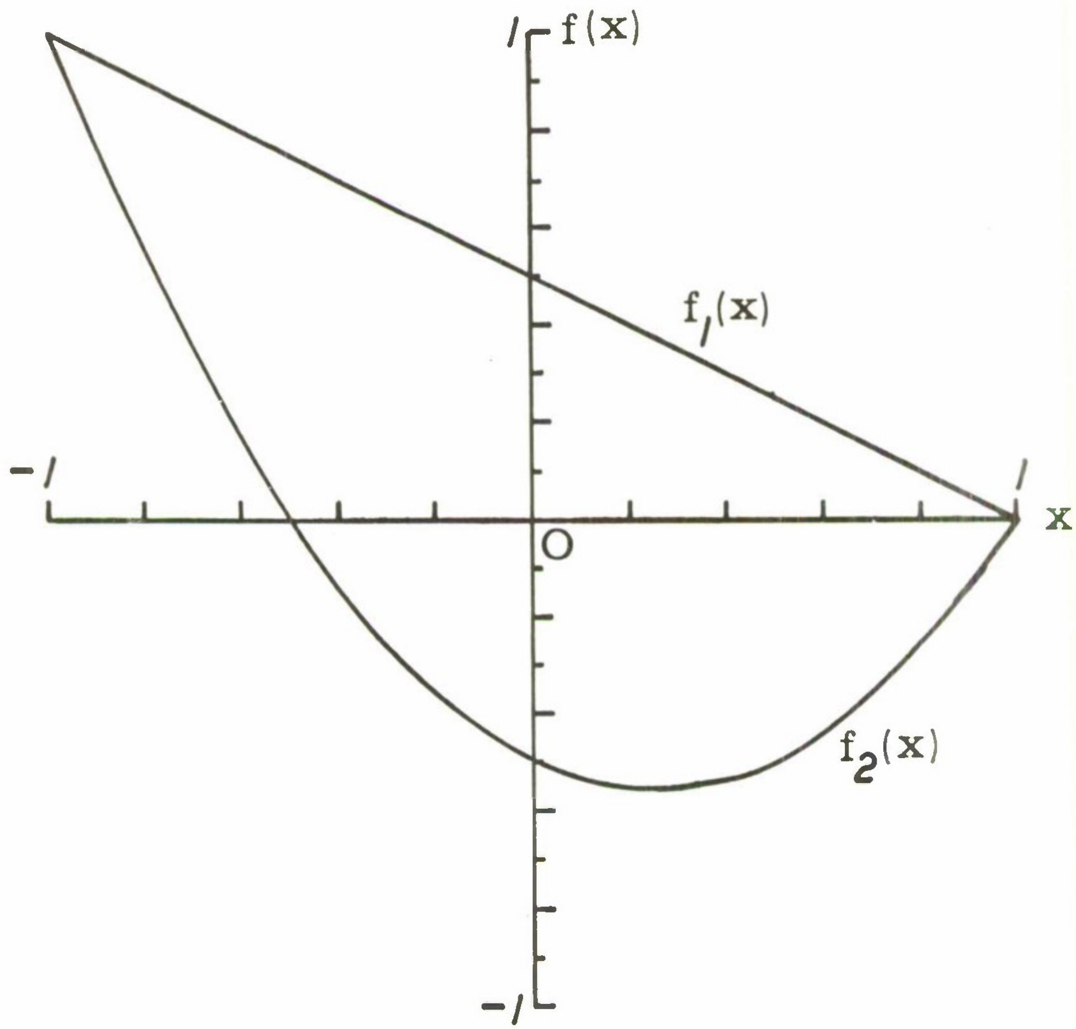

Figure 2. Variation of the coordinate functions with x.

Substitution of equation (30) into equation (10) leads to the following expression for the kinetic energy:

$$T = \frac{1}{2}\rho h \int_0^{\ell} [\frac{1}{3}(\dot{u}^2+\dot{w}^2) + \frac{1}{5}(\dot{\psi}^2 + \dot{\phi}^2)]\, dx_1. \tag{34}$$

By equations (7) and (30), one has

$$\varepsilon_{11} = f_1 u_{,1} + f_2 \psi_{,1} \qquad \varepsilon_{22} = f_{1,2}\, w + f_{2,2}\phi$$

$$\varepsilon_{12} = \frac{1}{2}(f_{1,2}u + f_{2,2}\psi + f_1 w_{,1} + f_2 \phi_{,1}),$$

so that

$$\int_A \sigma_{\alpha\beta}\varepsilon_{\alpha\beta} dA = \int_0^{\ell} [u_{,1}N_1 + \psi_{,1}N_2 + uQ_1 + \psi Q_2 + w_{,1}Q_3 + \phi_{,1}Q_4 +$$

$$+ wT_1 + \phi T_2]\, dx_1, \tag{35}$$

where the following "generalized stresses" have been defined:

$$N_1 = \int_{-h/2}^{h/2} f_1 \sigma_{11} dx_2, \quad N_2 = \int_{-h/2}^{h/2} f_2 \sigma_{11} dx_2,$$

$$Q_1 = \int_{-h/2}^{h/2} f_{1,2}\, \sigma_{12}\, dx_2, \quad Q_2 = \int_{-h/2}^{h/2} f_{2,2}\sigma_{12}\, dx_2,$$

$$Q_3 = \int_{-h/2}^{h/2} f_1 \sigma_{12} dx_2, \qquad Q_4 = \int_{-h/2}^{h/2} f_2 \sigma_{12} dx_2,$$

$$T_1 = \int_{-h/2}^{h/2} f_{1,2}\, \sigma_{22} dx_2, \qquad T_2 = \int_{-h/2}^{h/2} f_{2,2}\, \sigma_{22} dx_2.$$

Next, suppose that one defines "generalized strains" $\gamma_{11}, \alpha_{11}, \ldots, \alpha_{22}$ such that

$$e_{11} = f_1\gamma_{11} + f_2\,\alpha_{11}, \qquad e_{22} = f_{1,2}\,\gamma_{22} + f_{2,2}\alpha_{22},$$
$$2e_{12} = f_{1,2}\gamma_{12} + f_{2,2}\,\alpha_{12} + f_1\zeta_{12} + f_2\xi_{12}. \tag{36}$$

Then, one can easily show that

$$\int_A \sigma_{\alpha\beta}e_{\alpha\beta}dA = \int_0^{\ell} [\gamma_u N_1 + \alpha_{11}N_2 + \gamma_{12}Q_1 + \alpha_{12}Q_2 + \zeta_{12}Q_3 +$$
$$+ \xi_{12}Q_4 + \gamma_{22}T_1 + \alpha_{22}T_2]\,dx_1. \tag{37}$$

Upon writing equation (5) as

$$\sigma_{\alpha\beta} = 2\mu e_{\alpha\beta} + \gamma e_{\sigma\sigma}\delta_{\alpha\beta}$$

and substituting this into equation (13), it follows that

$$2U = c_1e^2_{11} + 2c_2e_{11}e_{22} + c_1e^2_{22} + 4\mu e^2_{12}, \tag{38}$$

where c_1 has been defined in equation (17) and

$$c_2 = 2\mu\lambda/(\lambda+2\mu) \tag{39}$$

by virtue of equation (6). In order to correct the shear deformation and transverse normal strain, correction factors κ_1 and κ_2 are introduced at this point. Specifically, e_{12} and e_{22} shall be replaced in equation (38) by κ_1e_{12} and κ_2e_{22}, respectively:

$$2U = c_1 e_{11}^2 + 2\kappa_2 c_2 e_{11} e_{22} + \kappa_2^2 c_1 e_{22}^2 + 4\kappa_1^2 \mu e_{12}^2. \qquad (40)$$

Inserting equation (36) into equation (40) and integrating the result over the domain $-h/2 < x_2 < h/2$, one obtains

$$\int_{-h/2}^{h/2} 2U\, dx_2 = (h/3)c_1\gamma_{11}^2 + (h/5)c_1\alpha_{11}^2 - \kappa_2 c_2 \gamma_{11}\gamma_{22} - \frac{7}{3}\kappa_2 c_2 \gamma_{11}\alpha_{22} +$$

$$+ \frac{1}{3}\kappa_2 c_2 \gamma_{22}\alpha_{11} - \kappa_2 c_2 \alpha_{11}\alpha_{22} + (1/h)\kappa_2^2 c_1 \gamma_{22}^2 +$$

$$+ (2/h)\kappa_2^2 c_1 \gamma_{22}\alpha_{22} + (19/3h)\kappa_2^2 c_1 \alpha_{22}^2 + (1/h)\kappa_1^2 \mu\gamma_{12}^2 +$$

$$+ (19/3h)\kappa_1^2 \mu\alpha_{12}^2 + (h/3)\kappa_1^2 \mu\zeta_{12}^2 + (h/5)\kappa_1^2 \mu\xi_{12}^2 +$$

$$+ (2/h)\kappa_1^2 \mu\gamma_{12}\alpha_{12} - \kappa_1^2 \mu\gamma_{12}\zeta_{12} + \frac{1}{3}\kappa_1^2 \mu\gamma_{12}\xi_{12} -$$

$$- \frac{7}{3}\kappa_1^2 \mu\alpha_{12}\zeta_{12} - \kappa_1^2 \mu\alpha_{12}\xi_{12}. \qquad (41)$$

Therefore, in view of equations (35), (37), and (41), equation (11) becomes

$$V = \int_0^{\ell} \{ u_{,1}N_1 + \psi_{,1}N_2 + uQ_1 + \psi Q_2 + w_{,1}Q_3 + \phi_{,1}\, Q_4 + wT_1 + \phi T_2 -$$

$$- \gamma_{11}N_1 - \alpha_{11}N_2 - \gamma_{12}Q_1 - \alpha_{12}Q_2 - \zeta_{12}Q_3 - \xi_{12}Q_4 - \gamma_{22}T_1 -$$

$$- \alpha_{22}T_2 + (h/6)c_1\gamma_{11}^2 + (h/10)c_1\alpha_{11}^2 - \frac{1}{2}\kappa_2 c_2\gamma_{11}\gamma_{22} -$$

$$- \frac{7}{6}\kappa_2 c_2 \gamma_{11}\alpha_{22} + \frac{1}{6}\kappa_2 c_2 \gamma_{22}\alpha_{11} - \frac{1}{2}\kappa_2 c_2 \alpha_{11}\alpha_{22} + (1/2h)\kappa_2^2 c_1 \gamma_{22}^2 +$$

$$+ (1/h)\kappa_2^2 c_1 \gamma_{22}\alpha_{22} + (19/6h)\kappa_2^2 c_1 \alpha_{22}^2 + (1/h)\kappa_1^2 \mu\gamma_{12}^2 +$$

$$+ (19/6h)\kappa_1^{\,2}\mu\alpha_{12}^{\,2} + (h/6)\kappa_1^{\,2}\mu\zeta_{12}^{\,2} + (h/10)\kappa_1^{\,2}\mu\xi_{12}^{\,2} + (2/h)\kappa_1^{\,2}\mu\gamma_{12}\alpha_{12} +$$

$$-\frac{1}{2}\kappa_1^{\,2}\mu\gamma_{12}\zeta_{12} + (1/6)\kappa_1^{\,2}\mu\gamma_{12}\xi_{12} - (7/6)\kappa_1^{\,2}\mu\alpha_{12}\zeta_{12} -$$

$$-\frac{1}{2}\kappa_1^{\,2}\mu\alpha_{12}\zeta_{12} - \frac{1}{2}\kappa_1^{\,2}\mu\alpha_{12}\xi_{12}\}\, dx_1. \tag{42}$$

Lastly, because tractions may be applied on the face $x_2 = -h/2$ of the plate and because of equation (2) and the fact that $f_\alpha(-h/2)=1$, equation (12) becomes simply

$$W = \int_0^\ell [p_1(x_1,t)(u+\psi) + p_2(x_1,t)\ (w+\phi)]\, dx_1, \tag{43}$$

under the assumption that either vanishing tractions or displacements are prescribed on the ends $x_1 = o, \ell$ of the plate.

The Lagrangian function L defined in equation (9) is now completely known because the required expressions for T, V, and W are given in equations (34), (42), and (43). Therefore, performing the variational process indicated in equation (8), one obtains the generalized stress equations of motion:

$$\begin{aligned}
&N_{1,1} - Q_1 + p_1 = \frac{1}{3}\rho h\ddot{u},\\
&N_{2,1} - Q_2 + p_1 = \frac{1}{5}\rho h\ddot{\psi},\\
&Q_{3,1} - T_1 + p_2 = \frac{1}{3}\rho h\ \ddot{w},\\
&Q_{4,1} - T_2 + p_2 = \frac{1}{5}\rho h\ddot{\phi},
\end{aligned} \tag{44}$$

which are subjected to one boundary condition from each of the following sets:

$$
\begin{aligned}
&\text{(i)} \quad \text{either } N_1 = 0 \text{ or } u = 0,\\
&\text{(ii)} \quad \text{either } N_2 = 0 \text{ or } \psi = 0,\\
&\text{(iii)} \quad \text{either } Q_3 = 0 \text{ or } w = 0,\\
&\text{(iv)} \quad \text{either } Q_4 = 0 \text{ or } \phi = 0
\end{aligned}
\tag{45}
$$

at $x_1 = 0$ and $x_1 = \ell$. The generalized stress-strain relations are

$$
\begin{aligned}
N_1 &= (h/3)c_1\gamma_{11} - \tfrac{1}{2}\kappa_2 c_2\gamma_{22} - \tfrac{7}{6}\kappa_2 c_2\alpha_{22},\\
N_2 &= (h/5)c_1\alpha_{11} + \tfrac{1}{6}\kappa_2 c_2\gamma_{22} - \tfrac{1}{2}\kappa_2 c_2\alpha_{22},\\
Q_1 &= (1/h)\kappa_1^2\mu\gamma_{12} + (2/h)\kappa_1^2\mu\alpha_{12} - \tfrac{1}{2}\kappa_1^2\mu\zeta_{12} + \tfrac{1}{6}\kappa_1^2\mu\xi_{12},\\
Q_2 &= (19/3h)\kappa_1^2\mu\alpha_{12} + (2/h)\kappa_1^2\mu\gamma_{12} - (7/6)\kappa_1^2\mu\zeta_{12} - \tfrac{1}{2}\kappa_1^2\mu\xi_{12},\\
Q_3 &= (h/3)\kappa_1^2\mu\zeta_{12} - \tfrac{1}{2}\kappa_1^2\mu\gamma_{12} - (7/6)\kappa_1^2\mu\alpha_{12},\\
Q_4 &= (h/5)\kappa_1^2\mu\xi_{12} + (\kappa_1^2/6)\mu\gamma_{12} - (\kappa_1^2/2)\mu\alpha_{12},\\
T_1 &= -\tfrac{1}{2}\kappa_2 c_2\gamma_{11} + \tfrac{1}{6}\kappa_2 c_2\alpha_{11} + (1/h)\kappa_2^2 c_1\gamma_{22} + (1/h)\kappa_2^2 c_1\alpha_{22},\\
T_2 &+ -\tfrac{7}{6}\kappa_2 c_2\gamma_{11} - \tfrac{1}{2}\kappa_2 c_2\alpha_{11} + (1/h)\kappa_2^2 c_1\gamma_{22} + (19/3h)c_1\kappa_2^2\alpha_{22},
\end{aligned}
\tag{46}
$$

and the generalized strain-displacement relations are

$$
\begin{aligned}
&\gamma_{11} = u_{,1} \qquad \alpha_{11} = \psi_{,1} \qquad \gamma_{12} = u, \qquad \alpha_{12} = \psi,\\
&\zeta_{12} = w_{,1} \qquad \xi_{12} = \phi_{,1} \qquad \gamma_{22} = w, \qquad \alpha_{22} = \phi.
\end{aligned}
\tag{47}
$$

5. THE FIRST ORDER THEORY

To obtain a first order theory, one sets $\psi = \phi = 0$, so that $\alpha_{\alpha\beta} = \xi_{12} = 0$ and equations (44) to (47) reduce to (a) the generalized equations of motion

$$\left.\begin{aligned} N_{1,1} - Q_1 + p_1 &= \frac{1}{3}\rho h\ddot{u}, \\ Q_{3,1} - T_1 + p_2 &= \frac{1}{3}\rho h\ddot{w}, \end{aligned}\right\} \quad 0<x_1<\ell, \tag{48}$$

(b) the boundary conditions

$$\begin{aligned} &\text{(i) either } N_1 = 0 \text{ or } u = 0, \\ &\text{(ii) either } Q_3 = 0 \text{ or } w = 0, \end{aligned} \tag{49}$$

at $x_1 = 0$ and $x_1 = \ell$, (c) the generalized stress-strain relations

$$\begin{aligned} N_1 &= (h/3)c_1\gamma_{11} - \frac{1}{2}\kappa_2 c_2\gamma_{22}, \\ Q_1 &= (1/h)\kappa_1^2\mu\gamma_{12} - \frac{1}{2}\kappa_1^2\mu\zeta_{12}, \\ Q_3 &= (h/3)\kappa_1^2\mu\zeta_{12} - \frac{1}{2}\kappa_1^2\mu\gamma_{12}, \\ T_1 &= -\frac{1}{2}\kappa_2 c_2\gamma_{11} + (1/h)\kappa_2^2 c_1\gamma_{22}, \end{aligned} \tag{50}$$

and (d) the generalized strain-displacement relations

$$\gamma_{11} = u_{,1} \qquad \gamma_{12} = u \qquad \zeta_{12} = w_{,1} \qquad \gamma_{22} = w. \tag{51}$$

Substitution of equation (51) into equation (50) yields expressions for the generalized stresses in terms of the generalized displacements:

$$N_1 = (h/3)\, c_1 u_{,1} - \frac{1}{2}\kappa_2 c_2 w,$$

$$Q_1 = (1/h)\kappa_1^{\;2}\mu u - \frac{1}{2}\kappa_1^{\;2}\mu w,_1 \quad ,$$

$$Q_3 = (h/3)\kappa_1^{\;2}\mu w,_1 - \frac{1}{2}\kappa_1^{\;2}\mu u, \tag{52}$$

$$T_1 = -\frac{1}{2}\kappa_2 c_2 u,_1 + (1/h)\kappa_2^{\;2} c_1 w,$$

where now $u(x_1,t)$ and $w(x_1,t)$ may be interpreted as the longitudinal and transverse deflections, respectively, of the face $x_2 = -h/2$ of the bar. The displacement form of the equations of motion can next be derived simply by inserting equation (52) into equation (48):

$$c_1 u,_{11} - (3/h^2)\kappa_1^{\;2}\mu u + (3/2h)(\kappa_1^{\;2}\mu - \kappa_2 c_2)w,_1 + (3/h)\, p_1 = \rho\ddot{u}, \tag{53}$$

$$\kappa_1^{\;2}\mu w,_{11} - (3/h^2)\kappa_2^{\;2} c_1 w + (3/2h)(\kappa_2 c_2 - \kappa_1^{\;2}\mu)u,_1 + (3/h)p_2 = \rho\ddot{w}. \tag{54}$$

To evaluate the correction factor κ_1 and κ_2 in equations (53) and (54), one possible approach consists of determining the lowest natural frequencies of vibration in the depth modes and comparing the results with equations (27) and (28). Hence, if one assumes that

$u = u(t)$, $w=w(t)$, and $p_1 = p_2 = 0$, then equations (53) and (54) become simply

$$\ddot{u} + \omega^2 u = 0, \qquad\qquad \ddot{w} + \omega^2 w = 0,$$

where, respectively,

$$\omega^2 = 3\kappa_1^{\;2}\mu/\rho h^2 \quad \text{and} \quad \omega^2 = 3\kappa_2^{\;2} c_1/\rho h^2.$$

Comparison of these expressions with equations (27) and (28) yields

$$\kappa_1^{\;2} = \kappa_2^{\;2} = \pi^2/12. \tag{55}$$

6. THE SECOND ORDER THEORY

The field equations for the second order theory are given in equations (44) to (47). If it is assumed that, for an infinitely long bar, $u = u(t), \ldots, \phi = \phi(t)$, then the deflection forms of the equations of motion are

$$\begin{aligned} &\ddot{u} + (3\kappa_1{}^2\mu/\rho h^2)\, u + (6\kappa_1{}^2\mu/\rho h^2)\psi = 0, \\ &(10\kappa_1{}^2\mu/\rho h^2)\, u + \ddot{\psi} + (95\kappa_1{}^2\mu/3\rho h^2)\psi = 0, \end{aligned} \tag{56}$$

for the depth-shear mode and

$$\begin{aligned} &\ddot{w} + (3\kappa_2{}^2 c_1/\rho h^2)\, w + (3\kappa_2{}^2 c_1/\rho h^2)\phi = 0, \\ &(5\kappa_2{}^2 c_1/\rho h^2)\, w + \ddot{\phi} + (95\kappa_2{}^2 c_1/3\rho h^2)\,\phi = 0, \end{aligned} \tag{57}$$

for the depth-stretch mode. If solutions of equations (56) and (57) are sought in the form

$$u(t) = A_1 \cos \omega t, \qquad \psi(t) = B_1 \cos \omega t,$$

$$w(t) = A_2 \cos \omega t, \qquad \phi(t) = B_2 \cos \omega t,$$

the following frequency equations are obtained in the usual way:

$$3(\omega/\omega_o)^4 - 104\,(\omega/\omega_o)^2 + 105 = 0, \qquad \omega_o{}^2 = \kappa_1{}^2\mu/\rho h^2, \tag{58}$$

for the depth-shear modes and

$$3(\omega/\omega_o)^4 - 104(\omega/\omega_o)^2 + 240 = 0, \quad \omega_o{}^2 = \kappa_2{}^2 c_1/\rho h^2, \tag{59}$$

for the depth-stretch modes.

The two solutions of the bi-quadratic equation (58) are

$$\omega_1^2 = 1.0408 \frac{\kappa_1^2 \mu}{\rho h^2}, \qquad \omega_2^2 = 33.6251 \frac{\kappa_1^2 \mu}{\rho h^2},$$

whereas, according to the expressions derived in Section 3, the exact results are

$$\omega_1^2 = \frac{\pi^2 \mu}{4\rho h^2}, \qquad \omega_2^2 = \frac{9\pi^2 \mu}{4\rho h^2}.$$

Thus, one might require that the values of ω_1^2 be identical, so that $\kappa_1^2 = \pi^2/4(1.0408)$ and

$$\omega_2 = 5.684 \frac{\pi}{2} \sqrt{\frac{\mu}{\rho h^2}},$$

which is <u>not</u> a good approximation to the exact expression

$$\omega_2 = 3 \frac{\pi}{2} \sqrt{\frac{\mu}{\rho h^2}}$$

For the depth-stretch modes, equation (59) yields

$$\omega_1^2 = 2.4859 \frac{\kappa_2^2 c_1}{\rho h^2}, \qquad \omega_2^2 = 32.1087 \frac{\kappa_2^2 c_1}{\rho h^2},$$

whereas the exact values are

$$\omega_1^2 = \frac{\pi^2}{4} \frac{c_1}{\rho h^2}, \qquad \omega_2^2 = \frac{9\pi^2}{4} \frac{c_1}{\rho h^2}$$

Proceeding as before, one finds $\kappa_2^2 = \pi^2/4(2.4859)$ and

$$\omega_2 = 3.594 \frac{\pi}{2} \sqrt{\frac{c_1}{\rho h^2}}$$

in contrast to the exact value

$$\omega_2 = 3\frac{\pi}{2}\sqrt{\frac{c_1}{\rho h^2}} \quad .$$

Clearly, the approximations for ω_2 in both the depth-shear and depth-stretch modes are not as accurate as might be desired. Indeed, it is evident from equations (56) and (57) that the u, ψ deflections are coupled in the depth-shear modes and the w, ϕ deflections are coupled in the depth-stretch modes. The fact is clearly at variance with the corresponding situation as embodied in equations (15) and (16) obtained from the "exact" theory of generalized plane stress. The source of this difficulty lies in the choice of coordinate functions $f_\alpha(x_2)$ made in equation (32) which do not satisfy the orthogonality condition

$$\int_{-h/2}^{h/2} f_{1,2}(x_2) f_{2,2}(x_2)\, dx_2 = 0 \tag{60}$$

which would eliminate all the undesirable coupling terms in equations (56) and (57). Additional comments on this point are given in Section 9.

7. FREE VIBRATIONS OF A CONSTRAINED BAR

Suppose that $p_1 = p_2 = 0$ and that $\kappa_\alpha^2 = \kappa^2$, where, according to equation (55), $\kappa^2 = \pi^2/12$. Furthermore, suppose that the changes of variables

$$x_1 = \ell x, \quad o<x<1, \qquad t = c\tau, \qquad c^2 = \rho\ell^2/\kappa^2\mu$$

are introduced into equations (53) and (54). Consequently, the dimensionless form of the equations of free motion of a constrained bar

are, according to the first order theory,

$$\left.\begin{aligned} \beta u'' - 3\xi^2 u + \gamma w' &= \ddot{u}, \\ w'' - 3\alpha\xi^2 w - \gamma u' &= \ddot{w}, \end{aligned}\right\} \quad o<x<1, \tag{61}$$

where $u' = \partial u/\partial x$, $\dot{u} = \partial u/\partial \tau$, etc. and

$$\beta = \alpha/\kappa^2, \quad \alpha = c_1/\mu, \quad \xi = \ell/h, \quad \gamma = (3\ell/2h)(\kappa\mu - c_2)/\kappa\mu \tag{62}$$

Suppose, in addition, that the ends $x_1 = 0$, ℓ of the beam are simply supported, i.e.,

$$u = Q_3 = 0 \text{ at } x_1 = 0, \ell.$$

In view of equation (52), these boundary conditions become effectively

$$u = w' = 0 \quad \text{at } x = 0, 1. \tag{63}$$

It is easily verified that the expressions

$$u(x,\tau) = A \sin n\pi x \cos \omega\tau, \; w(x,\tau) = B \cos n\pi x \cos \omega\tau, \tag{64}$$

where $n = 1,2,3,\ldots$, A and B are arbitrary constants, and ω is the dimensionless circular frequency parameter, satisfy both the boundary conditions in equation (63). Insertion of equation (64) into equation (61) yields the system of homogeneous algebraic equations

$$[\omega^2 - \beta(n\pi)^2 - 3\xi^2]\, A - n\pi\gamma B = 0,$$

$$-n\pi\gamma A + [\omega^2 - (n\pi)^2 - 3\alpha\xi^2]\, B = 0,$$

which has non-trivial solutions if and only if

$$\begin{vmatrix} \omega^2 - \beta(n\pi)^2 - 3\xi^2 & -n\pi\gamma \\ -n\pi\gamma & \omega^2 - (n\pi)^2 - 3\alpha\xi^2 \end{vmatrix} = 0.$$

Expansion of this determinant yields the (dimensionless)frequency equation

$$\omega^4 - [(1+\beta)\,(n\pi)^2 + 3(1+\alpha)\,\xi^2]\,\omega^2 + \beta(n\pi)^4 +$$

$$+ [3(1+\alpha\beta)\xi^2 - \gamma^2]\,(n\pi)^2 + 9\alpha\xi^4 = 0,$$

which has the solutions

$$(\omega_{j,n})^2 = \frac{1}{2}\,[(1+\beta)(n\pi)^2 + 3(1+\alpha)\xi^2] + \frac{1}{2}\,(-1)^j \Big\{[(1-\beta)(n\pi)^2 -$$

$$-3(1-\alpha)\xi^2]^2 + (2n\pi\gamma)^2 \Big\}^{1/2}, \quad j = 1,2, \qquad (65)$$

where $\omega_{1,n}$ and $\omega_{2,n}$ denote the (dimensionless) natural frequencies of the transverse and longitudinal modes of vibration, respectively. By virtue of equations (17), (39), and (55), the quantities defined in equation (62) can be expressed as

$$\alpha = 1/8\,(1-\nu),\ \beta = 3/2\pi^2(1-\nu),\ \gamma = (3\xi/2)\,[1-4\nu\sqrt{3}/\pi(1-\nu)],$$

where ν denotes Poisson's ratio.

For large values of the length-to-depth ratio ξ, one can show that the $\omega_{j,n}^2$ behave asymptotically as

$$\omega_{1,n}^2 = 3\alpha\xi^2 + \frac{(n\pi)^2}{1-\alpha}\left\{1-\alpha-\frac{3}{4}\left[1-\frac{2\nu}{\kappa(1-\nu)}\right]^2\right\} + O(\xi^{-2}) \qquad (66)$$

for the transverse modes and

$$\omega_{2,n}^{2} = 3\xi^{2} + (n\pi)^{2}\left\{\beta + \frac{3}{4(1-\alpha)}\left[1 - \frac{2\nu}{\kappa(1-\nu)}\right]^{2}\right\} + 0(\xi^{-2}) \qquad (67)$$

for the longitudinal modes. As $\xi\to\infty$, equations (66) and (67) become

$$\omega_{1,n} \sim \xi\sqrt{3\alpha} \qquad \text{and} \qquad \omega_{2,n} \sim \xi\sqrt{3}\,, \qquad (68)$$

which can be shown to be the dimensionless forms of the fundamental frequencies of the depth-stretch and depth-shear modes, respectively, in an infinite, constrained plate strip (see Section 3).

In Figures 3 and 4, the variations of $\omega_{j,n}$ have been plotted against ξ for $\nu = 0.3$ and $n = 1(1)8$. These curves show, for sufficiently large values of ξ, the linear rate of growth in frequency as indicated by equation (68). For $j = 1$ and 2, the frequency of the lowest mode, $n = 1$, is virtually a straight line in the $\xi\omega$-plane. As n increases, in the case of $j=1$, the frequency curves rise steeply with increasing ξ and then abruptly change slope, growing thereafter at a slower rate. In the second mode ($j=2$), the higher frequencies increase with ξ very slowly until an abrupt increase in slope is observed. The point of this transition in slope occurs for larger and larger values of ξ as the mode number n is increased.

8. BEAMS ON AN ELASTIC FOUNDATION

The classical equations of extensional and flexural motion of slender elastic beams of length ℓ_1 and rectangular cross section with depth H and thickness b are, respectively,

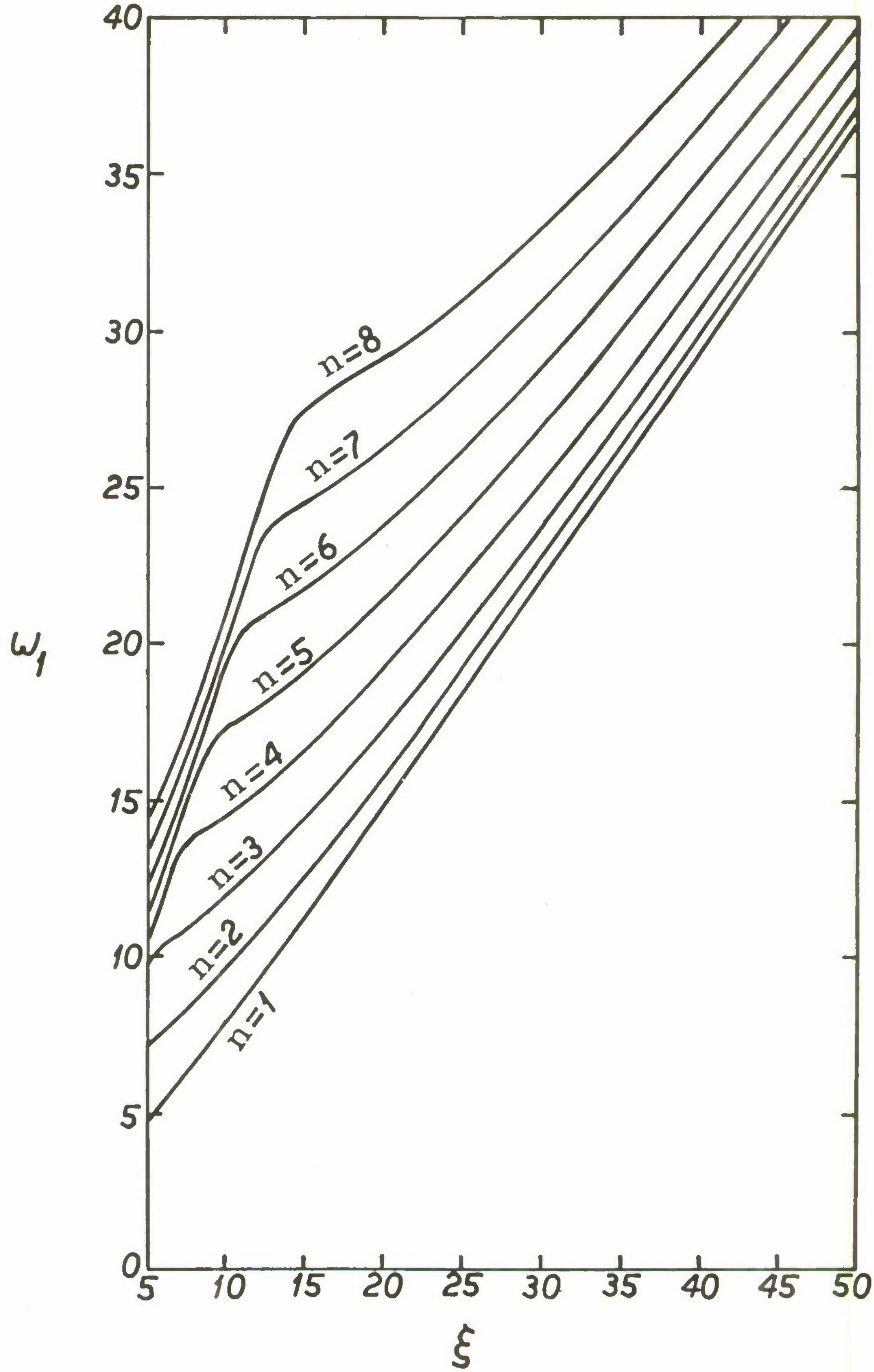

Figure 3. Variation of the transverse natural frequency ω_1 versus ξ for the first eight modes.

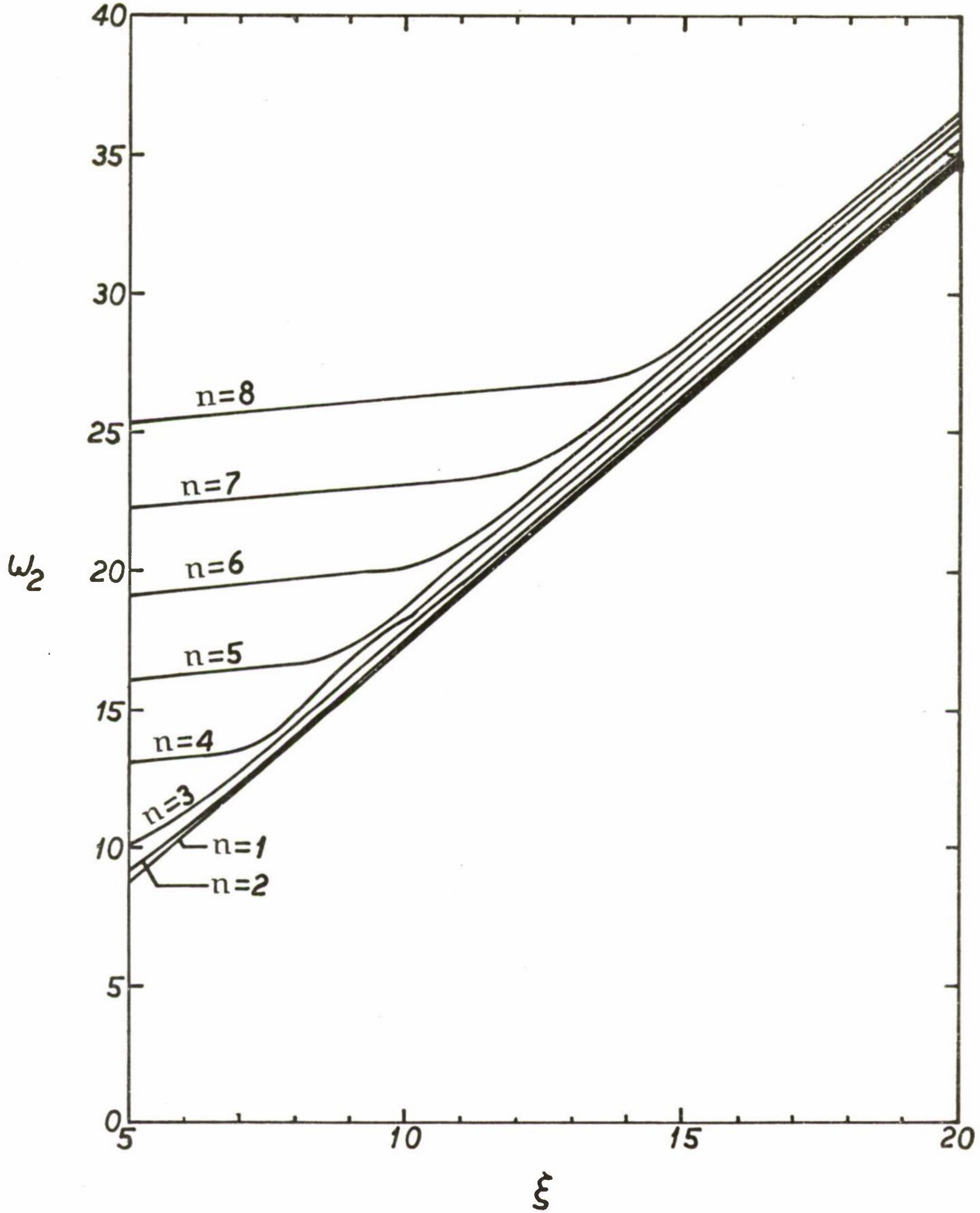

Figure 4. Variation of the longitudinal natural frequency ω_2 versus ξ for the first eight modes.

$$E A \bar{u}_{,11} + b\,[\sigma_{12}^{(+)} - \sigma_{12}^{(-)}] = \rho_b\, A\, \ddot{\bar{u}} \tag{69}$$

and

$$E I \bar{w}_{,1111} + \rho_b A \ddot{\bar{w}} = b[\sigma_{22}^{(+)} - \sigma_{22}^{(-)}] + \frac{1}{2} A[\sigma_{12,1}^{(+)} + \sigma_{12,1}^{(-)}], \tag{70}$$

where $\sigma_{\alpha 2}^{(\pm)} = \sigma_{\alpha 2}\,|_{x_2 = \pm H/2}$, $A = bH$, $I = bH^3/12$, E denotes Young's modulus, ρ_b the mass density of the beam, $\bar{u}\,(x_1,t)$ the longitudinal deflection and $\bar{w}\,(x_1,t)$ the transverse deflection of the center line of the beam. These deflections are related to the displacements $\bar{u}_1\,(x_1,x_2,t)$ and $\bar{u}_2(x_1,x_2,t)$ according to

$$\bar{u}_1\,(x_1,x_2,t) = \bar{u}(x_1,t) - x_2 \bar{w}_{,1}\,(x_1,t), \tag{71}$$

$$\bar{u}_2\,(x_1,x_2,t) = \bar{w}\,(x_1,t),$$

for $o<x_1<\ell_1$, $-H/2<x_2<H/2$.

Under the hypothesis that the beam is perfectly bonded to the elastic foundation at the interface, the longitudinal and transverse deflections must be equal there, i.e.,

$$\bar{u}_\alpha\,(x_1,\ H/2,\ t) = u_\alpha\,(x_1,\ -h/2,\ t),$$

which, in view of equations (30), (32), and (71), yield

$$\left.\begin{aligned} \bar{u}\,(x_1,t) - (H/2)\,\bar{w}_{,1}\,(x_1,t) &= u(x_1,t) \\ \bar{w}\,(x_1,t) &= w\,(x_1,t) \end{aligned}\right\} \tag{72}$$

for the first order foundation theory. In addition, at the interface one also has $\sigma_{\alpha 2}^{(+)} = -p_\alpha$, where, by virtue of equations (53) and (54),

$$\sigma_{12}^{(+)} = (\rho h/3)\ddot{u} + (c_1 h/3)u,_{11} - (\kappa^2\mu/h)u + (\kappa/2)(\kappa\mu - c_2)w,_1 \tag{73}$$

$$\sigma_{22}^{(+)} = -(\rho h/3)\ddot{w} + (\kappa^2\mu h/3)w,_{11} - (\kappa^2 c_2/h)w + (\kappa/2)(c_2 - \kappa\mu)u,_1. \tag{74}$$

Some special cases are of interest. As a first example, suppose that the beam is loaded such that extensional deformations predominate over flexural deformations. It might then be assumed that $w = \overline{w} \approx 0$, so that the pertinent equations of motion for the beam and the foundation are equation (69) and

$$\sigma_{12}^{(+)} = -(\rho h/3)\ddot{u} + (c_1 h/3)u,_{11} - (\kappa^2\mu/h)\, u. \tag{75}$$

Substitution of equation (75) into equation (69) and use of $\overline{u}(x_1,t) \approx u(x_1,t)$ yield

$$(EA + bhc_1/3)u,_{11} - (\kappa^2 b\mu/h)u = (\rho_b A + \rho bh/3)\ddot{u} + b\sigma_{12}^{(-)}. \tag{76}$$

As a second example, suppose that the beam is loaded in such a manner that transverse deformations predominate over extensional deformations. This suggests that, as a first approximation, $u = \overline{u} = 0$ and $w = \overline{w}$. In particular, equation (74) may be approximated by

$$\sigma_{22}^{(+)} = -(\rho h/3)\ddot{w} + (\kappa^2\mu h/3)\, w,_{11} - (\kappa^2 c_2/h)w. \tag{77}$$

Insertion of equation (77) into equation (70), under the hypothesis that $\sigma_{12}^{(\pm)} = 0$, leads to

$$EIw,_{1111} - (bh\kappa^2\mu/3)w,_{11} + (b\kappa^2 c_2/h)w +$$
$$+ (\rho_b A + \rho bh/3)\ddot{w} = -b\sigma_{22}^{(-)}. \tag{78}$$

Equation (78) possesses a form that is quite similar to the one investigated in references [3], [5], and [9] to [11]. As cited in equation (1), the term $-(\kappa^2 c_2/h)w$ in equation (77) is analogous to the Winkler hypothesis, whereas the set of terms $(\kappa^2\mu h/3)w,_{11} - (\kappa^2 c_2/h)w$ is sometimes identified as either the Wieghardt or Pasternak model for an elastic foundation. If the inertia term $-(\rho h/3)\ddot{w}$ is included with these latter terms, one may speak of a Wieghardt- or Pasternak-type inertial foundation. Essentially, the new feature in equation (78) consists of the introduction of the correction factor $\kappa^2 = \pi^2/12$, which resulted upon the insistence that the foundation model should produce the fundamental depth-stretch frequency for an infinitely long bar.

In the general coupled case, one finds, upon inserting equation (72) into equations (73) and (74),

$$\sigma_{12}^{(+)} = -(\rho h/3)\ddot{\bar{u}} + (c_1 h/3)\bar{u},_{11} - (\kappa^2\mu/h)\bar{u} + (\rho hH/6)\ddot{\bar{w}},_1 -$$

$$- (c_1 hH/6)\bar{w},_{111} + (\kappa/2h)\,[\kappa\mu(H+h) - hc_2]\bar{w},_1, \tag{79}$$

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$$\sigma_{22}{}^{(+)} = -(\rho h/3)\ddot{\bar{w}} + (\kappa/12)[\kappa\mu(4h+3H)-3HC_2]\bar{w}_{,11} -$$

$$-(\kappa^2 c_2/h)\bar{w} + (\kappa/2)(c_2 - \kappa\mu)\bar{u}_{,1}. \qquad (80)$$

Consequently, substitution of equations (79) and (80) into equations (69) and (70) yields

$$(EH+c_1 h/3)\,\bar{u}_{,11} - (\kappa^2\mu/h)\bar{u} - (c_1 hH/6)\bar{w}_{,111} +$$

$$+ (\kappa/2h)[\kappa\mu(H+h)-hc_2]\,\bar{w}_{,1} = (\rho_b H + \rho h/3)\,\ddot{\bar{u}} -$$

$$-(\rho hH/6)\,\ddot{\bar{w}}_{,1} + \sigma_{12}{}^{(-)}, \qquad (81)$$

and

$$-(c_1 bhH/6)\bar{u}_{,111} + (\kappa b/2h)\,[\kappa\mu(H+h) - hc_2]\,\bar{u}_{,1} +$$

$$+ (EI + c_1 bhH^2/12)\,\bar{w}_{,1111} - (\kappa b/12h)[\kappa\mu(H+2h)^2 - 4Hhc_2]\,\bar{w}_{,11} +$$

$$+ (\kappa^2 c_2 b/h)\,\bar{w} = -(\rho bhH/6)\,\ddot{\bar{u}}_{,1} - (\rho_b bH + \rho bh/3)\,\ddot{\bar{w}} +$$

$$+ (\rho hbH^2/12)\,\ddot{\bar{w}}_{,11} - b\,\sigma_{22}{}^{(-)} + \tfrac{1}{2}bH\,\sigma_{12,1}{}^{(-)}. \qquad (82)$$

Neglecting the rotatory inertia terms $\ddot{\bar{w}}_{,1}$ and $\ddot{\bar{w}}_{,11}$ as well as $\ddot{\bar{u}}_{,1}$ in equations (81) and (82), one obtains

$$(EH + c_1 h/3)\bar{u}_{,11} - (\kappa^2\mu/h)\,\bar{u} - (c_1 hH/6)\bar{w}_{,111} +$$

$$+ (\kappa/2h)[\kappa\mu(H+h)-hc_2]\bar{w}_{,1} = (\rho_b H + \rho h13)\ddot{\bar{u}} + \sigma_{12}{}^{(-)}, \qquad (83)$$

and

$$-(c_1 bhH/6)\bar{u}_{,111} + (\kappa b/2h)[\kappa\mu(H+h) - hc_2]\bar{u}_{,1} +$$

$$+ (EI+c_1 bhH^2/12)\overline{w}_{,1111} - (\kappa b/12h)[\kappa\mu(H+2h)^2 - 4Hhc_2]\,\overline{w}_{,11} +$$

$$+ (\kappa^2 c_2 b/h)\overline{w} = -(\rho_b bH+\rho bh/3)\,\ddot{\overline{w}} - b\sigma_{22}^{(-)} + \frac{1}{2}\,bH\sigma_{12,1}^{(-)}. \qquad (84)$$

9. CONCLUSIONS

In Section 6, it was observed that the second order theory does not provide accurate approximations for the second depth-shear and depth-stretch frequencies in an infinite plate strip. This difficulty was traced to the fact that the selected coordinate functions $f_\alpha(x_2)$ stated in equation (32) do not satisfy the orthogonality condition presented in equation (60).

The undesirable coupled terms in the equations of motion can be eliminated upon selecting the coordinate functions $f_\alpha(x_2)$ to be

$$\left.\begin{aligned} f_1(x_2) &= (h-2x_2)/2h, \\ f_2(x_2) &= (h-2x_2)\,(h+2x_2)\,(h+10x_2)/h^3, \end{aligned}\right\} \qquad (85)$$

where these functions satisfy the orthogonality conditions (33) and (60). If a second order theory is required, the theory can be reformulated following the procedure outlined in Section 4, provided that one now use the coordinate functions given in equation (85).

On the other hand, instead of using these polynomial coordinate functions, it seems still more desirable to employ the eigenfunctions given in equation (29) as the coordinate functions. Specifically, higher order theories for constrained bars could be based upon expansions of the form

$$u_\alpha(x_1,x_2,t) = \sum_{n=1}^{\infty} u_\alpha^{(n)}(x_1,t)\,\phi^{(n)}(x_2),$$

where, in view of equation (29), one might select

$$\phi^{(n)}(x_2) = (-1)^n \sin[(2n-1)\,\pi(h-2x_2)/4h], \quad -h/2<x_2<h/2,$$

as coordinate functions. These functions satisfy the orthogonality conditions

$$\int_{-h/2}^{h/2} \phi^{(n)}\phi^{(k)}\,dx_2 = 0 \quad \text{if } n \neq k$$

and

$$\int_{-h/2}^{h/2} \phi_{,2}^{(n)}\,\phi_{,2}^{(k)}\,dx_2 = 0 \text{ if } n \neq k.$$

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